ENGINEERING MATH - II

UNIT 5

FOURIER SERIES

Vibha Masti

Feedback/corrections: vibha@pesu.pes.edu

Fourier series

- Jacob Fourier
- Series expansion used for periodic signals to expand in terms of their harmonics which are sinusoidal or orthogonal to one another.

Fourier Representation

- Fourier series — periodic signals
  - Continuous time — FS
  - Discrete time — DTFS
- Fourier transforms — aperiodic signals
  - Continuous time — FT
  - Discrete time — DTFT

Continuous time FS

1. Trignometric FS
2. Complex exponential FS

Periodic Function

$$f(x+T) = f(x) \quad \forall \; x \in \mathbb{R}$$

(T is +ve → period)

Fundametal Period (smallest period)

Properties of Periodic Functions

1. If T is period of $f(x)$, nT is also period $(n \in \mathbb{Z})$

2. If $f(x)$ & $g(x)$ have periods T, then

$$h(x) = a f(x) + b g(x) \text{ has period } T$$

3. If $f(x)$ is periodic with period T, then $f(ax)$ is periodic with period $\frac{T}{a}$.

4. Period of sum of periodic functions is LCM of periods

5. constants are periodic for any period T

(I) TRIGONOMETRIC SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n are called coefficients

Fourier - Euler Formulas

for coefficients

let $f(x)$, a periodic function of $T = 2\pi$ be defined in the interval $(\alpha, \alpha + 2\pi)$ as the sum of a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \quad \text{——— (1)}$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \quad \text{for } n = 1, 2, 3 \ldots \text{ — (2)}$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \quad \text{for } n = 1, 2, 3 \ldots \text{ — (3)}$$

$n = 1 \longrightarrow$ First harmonic

$n = 2 \longrightarrow$ Second harmonic

⋮

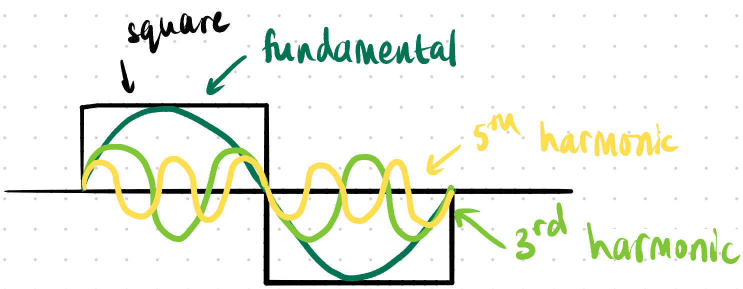
Dirichlet Conditions

Let $f(x)$ be periodic function with $T = 2\pi$. Let $f(x)$ be a piecewise continuous function in the interval $(\alpha, \alpha+2\pi)$ with finite number of extrema. Then

1. At the point of continuity, Fourier series of $f(x)$ RHS converges to $f(x)$ LHS

2. At the point of discontinuity, Fourier series of $f(x)$ converges to arithmatic mean of left and right hand limits of $f(x)$.

Harmonics



1. Obtain FS to represent e^{-ax} from $x=-\pi$ to $x=\pi$ and hence derive series for $\frac{\pi}{\sinh\pi}$

$$f(x) = e^{-ax}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{-a} + \frac{e^{a\pi}}{a}\right] = \left(\frac{1}{\pi a}\right)2\sinh a\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-e^{-a\pi}}{a^2+n^2}\left((-1)^n a\right) + \frac{e^{a\pi}}{a^2+n^2}\left((-1)^n a\right)\right.$$

$$a_n = \frac{1}{\pi}\left(\frac{(-1)^n a}{a^2+n^2}\right)2\sinh a\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}\left((-1)^{n+1} n\right) - \frac{e^{a\pi}}{a^2+n^2}\left((-1)^{n+1} n\right)\right]$$

$$b_n = \frac{1}{\pi}\left(\frac{(-1)^n n}{a^2+n^2}\right) 2\sinh a\pi$$

$$f(x) = e^{-ax} = \left(\frac{\sinh a\pi}{a}\right)\left(\frac{1}{\pi}\right) +$$

$$\sum_{n=1}^{\infty} \frac{1}{\pi}\left(2\sinh a\pi\right)\left(\frac{(-1)^n}{a^2+n^2}\right)\left(a\cos nx + n\sin nx\right)$$

when $x=0$ & $a=1$

$$1 = \left(\frac{\sinh \pi}{1}\right)\left(\frac{1}{\pi}\right) + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(2\sinh\pi\right)\frac{(-1)^n}{1+n^2}$$

$$1 = \left(\frac{\sinh \pi}{1}\right)\left(\frac{1}{\pi}\right)\left(1 + 2\left(\frac{-1}{1+1^2} + \frac{1}{1+2^2} \frac{-1}{1+3^2}\right)\right)$$

$$\frac{\pi}{\sinh \pi} = 2\left(\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} \cdots\right)$$

2. Obtain FS of $f(x) = x + x^2$ in $(-\pi, \pi)$. Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2 \, dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3} \quad \Rightarrow \quad \frac{a_0}{2} = \frac{\pi^2}{3}$$

odd even even

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \, dx$$

$u = x^2 \qquad v = \frac{\sin nx}{n}$

$du = 2x \qquad dv = \cos nx$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \int_0^\pi \frac{2x}{n} \sin nx \, dx \right]_0^\pi$$

$$= \frac{2}{\pi}\left(0 - \int_0^\pi \frac{2x}{n} \sin nx \, dx \right)$$

$$= \frac{-2}{\pi}\left(\frac{2}{n}\right) \int_0^\pi x \sin nx \, dx$$

$u = x \qquad v = -\frac{\cos nx}{n}$

$du = dx \qquad dv = \sin nx$

$$= \frac{-4}{\pi n}\left[\frac{-x \cos nx}{n} + \int \frac{\cos nx}{n} \, dx \right]_0^\pi$$

$$= \frac{4(-1)}{n\cancel{\pi}}\left((-1)^{n+1} \frac{\cancel{\pi}}{n} + \left[\frac{\sin nx}{n^2} \right]_0^\pi \right)$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2) \sin nx \, dx$$

odd even odd

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$u = x$ $du = dx$

$v = \frac{-\cos nx}{n}$ $dv = \sin nx \, dx$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \int \frac{\cos nx}{n} dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n}(-\pi)\right)$$

$$= \frac{-2(-1)^n}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$(x+x^2) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin nx$$

Function defined from $(-\pi, \pi)$

$x = \pi$ (point of discontinuity) Dirichlet condition

$f(\pi)$ = arithmatic mean of LHL & RHL

RHL at π:

function is periodic with period 2π $\Rightarrow$ $f(\pi^+) = f(-\pi^+)$

$$f(\pi^+) = f(-\pi^+) = -\pi + \pi^2$$

LHL at π:

$$f(\pi^-) = \pi + \pi^2$$

$$\therefore f(\pi) = \frac{1}{2}\left(f(\pi^+) + f(\pi^-)\right)$$

$$= \frac{1}{2}\left(\pi^2 + \pi - \pi + \pi^2\right) = \pi^2$$

$$\therefore f(\pi) = \pi^2$$

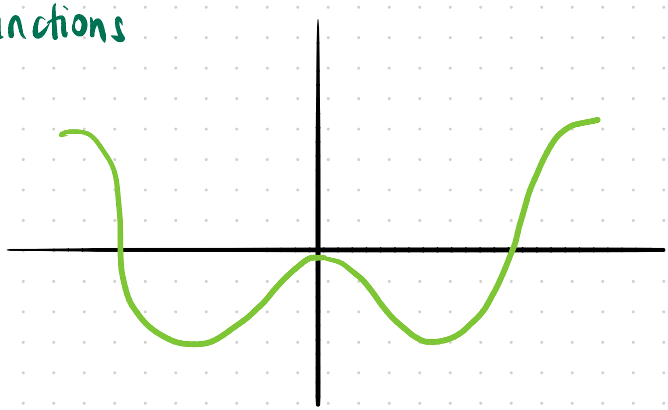
$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos n\pi + 0$$

$$\frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} (-1)^n (-1)^n \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Odd & Even Functions

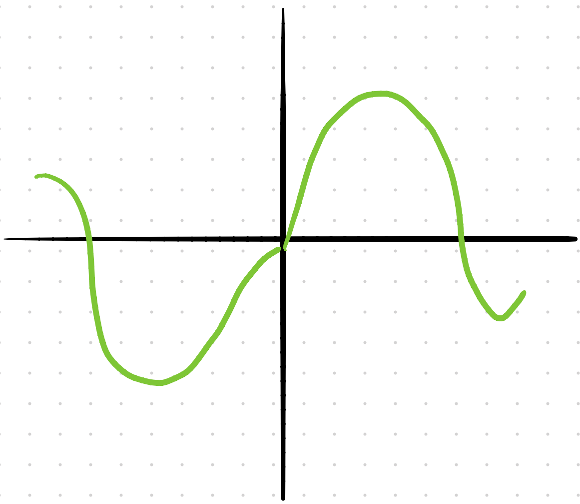
Even Functions



$f(x) = f(-x)$

can be represented as cosine waves

Odd Functions



$f(-x) = -f(x)$

can be represented as sine waves

Fourier Series Expansion of Even Function

If $f(x)$ is an even function in $(-\pi, \pi)$, then the Fourier series expansion contains 'only' cosine terms

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

Fourier Series Expansion of Odd Function

If $f(x)$ is an odd function in $(-\pi, \pi)$, then the Fourier series expansion contains 'only' sine terms

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

3. Find FS of $f(x) = \begin{cases} x(\pi - x) & -\pi < x \le 0 \\ x(\pi + x) & 0 < x < \pi \end{cases}$ $\phi_1(x)$, $\phi_2(x)$

Check for parity:

$$\phi_1(-x) = (-x)(\pi + x) = -\phi_2(x)$$

$$\phi_2(-x) = -x(\pi - x) = -\phi_1(x)$$

$\therefore$ odd function

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

odd even odd

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} (x\pi + x^2)\sin nx\,dx$$

$$b_n = \frac{2}{\cancel{\pi}}\int_0^{\pi} \cancel{\pi} x \sin nx\,dx = 2\int_0^{\pi} x \sin nx\,dx$$

$u = x$ $v = \frac{-\cos nx}{n}$

$du = dx$ $dv = \sin nx\,dx$

$$= 2\left[\frac{-x\cos nx}{n} + \int \frac{\cos nx}{n} \right]_0^{\pi}$$

$$= 2\left(\frac{-\pi(-1)^n}{n} \right) = \frac{-2\pi(-1)^n}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2\pi(-1)^n}{n} \sin nx$$

$$f(x) = -2\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Fourier Series of Functions with Period 2L

Let $f(x)$ be a periodic function with arbitrary period $2L$ defined in an interval $c < x < c+2L$. Then the Fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{n} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

For Even Functions

For an even function f(x) in (-L, L), FS contains only cosine terms

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

For Odd Functions

For an even function f(x) in (-L, L), FS contains only cosine terms

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

4. Find FS expansion of $f(x) = x(1-x)(2-x)$ in $(0,2)$.

Deduce the sum of the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$

$$f(x) = (x-x^2)(2-x) = 2x - 2x^2 - x^2 + x^3$$

$$= x^3 - 3x^2 + 2x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{1}\int_0^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^2$$

$$= 4 - 8 + 4 = 0$$

$$a_n = \frac{1}{1}\int_0^2 \underbrace{(x^3 - 3x^2 + 2x)}_{u}\,\underbrace{\cos n\pi x\,dx}_{dv}$$

$$= \left[(x^3 - 3x^2 + 2x)\left(\frac{\sin n\pi x}{n\pi}\right) - \int \underbrace{(3x^2 - 6x + 2)}_{u}\,\underbrace{\left(\frac{\sin n\pi x}{n\pi}\right) dx}_{dv}\right]$$

$$= \left[(x^3 - 3x^2 + 2x)\left(\frac{\sin n\pi x}{n\pi}\right) + (3x^2 - 6x + 2)\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. - \int \underbrace{(6x - 6)}_{u}\,\underbrace{\left(\frac{\cos n\pi x}{n^2\pi^2}\right)}_{dv} dx\right]_0^2$$

$$= \left[(x^3-3x^2+2x)\left(\frac{\sin n\pi x}{n\pi}\right) - (3x^2-6x+2)\frac{(\cos n\pi x)}{(n\pi)^2} - (6x-6)\frac{\sin n\pi x}{(n\pi)^3} - 6\frac{\cos n\pi x}{(n\pi)^4} \right]_0^2$$

$$= -(3\times 4 - 6\times 2 + 2)\frac{\cos 2n\pi}{(n\pi)^2} - \frac{6}{(n\pi)^4} + \frac{2}{(n\pi)^2} + \frac{6}{(n\pi)^4}$$

$$= \frac{-2}{(n\pi)^2} - \frac{6}{(n\pi)^4} + \frac{2}{(n\pi)^2} + \frac{6}{(n\pi)^4} = 0$$

$a_n = 0$

OR

$f(x) = x(1-x)(2-x) \qquad [0,2]$

$f(2-x) = (2-x)(1-2+x)(\cancel{2}-\cancel{2}+x)$

$= -(x-1)(x-2)(x) = -x(1-x)(2-x)$

$\therefore f(2-x) = -f(x) \Rightarrow$ ODD FUNCTION

$a_0 = a_n = 0$

$$b_n = \frac{2}{1}\int_0^1 f(x) \sin \frac{n\pi x}{L}\, dx$$

$$b_n = 2\int_0^1 \underbrace{(x^3-3x^2+2x)}_{u}\ \underbrace{\sin n\pi x}_{dv}\ dx$$

$$= 2\left[-(x^3-3x^2+2x)\frac{\cos n\pi x}{n\pi} + \int \underbrace{\frac{\cos n\pi x}{n\pi}}_{dv}\underbrace{(3x^2-6x+2)}_{u}\,dx\right]_0^1$$

$$= 2\left[-(x^3-3x^2+2x)\frac{\cos n\pi x}{n\pi} + (3x^2-6x+2)\frac{\sin n\pi x}{(n\pi)^2}\right.$$

$$\left. - \int \underbrace{(6x-6)}_{u}\ \underbrace{\frac{\sin n\pi x}{(n\pi)^2}}_{dv}\ dx\right]_0^1$$

$$= 2\left[-(x^3-3x^2+2x)\frac{\cos n\pi x}{n\pi} + (3x^2-6x+2)\frac{\sin n\pi x}{(n\pi)^2}\right.$$

$$\left. + (6x-6)\frac{\cos n\pi x}{(n\pi)^3} + 6\,\frac{\sin n\pi x}{(n\pi)^4}\right]_0^1$$

$$b_n = 2\left[6\,\frac{(1)}{(n\pi)^3}\right] = \frac{12}{(n\pi)^3}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{12}{(n\pi)^3}\sin n\pi x$$

$$x^3-3x^2+2x = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin n\pi x$$

Let $x = 1/2$

$$\frac{1}{8} - \frac{3}{4} + 1 = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sin \frac{n\pi}{2}\right)$$

$$\frac{\cancel{3}\pi^3}{8 \times \cancel{12}4} = \frac{\pi^3}{32} = \left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \cdots\right)$$

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \cdots = \frac{\pi^3}{32}$$

Parseval's Theorem

$$\int_{-L}^{L} [f(x)]^2 \, dx = L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

Provided the Fourier Series of $f(x)$ converges in $(-L, L)$

Different Cases

(1) $f(x)$ is even

$b_n = 0$

$$2\int_{0}^{L} [f(x)]^2 dx = L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$\int_{0}^{L} [f(x)]^2 dx = \frac{L}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$$

(2) $f(x)$ is odd

$$a_0 = 0, \quad a_n = 0$$

$$2\int_0^L [f(x)]^2 dx = L\left[\sum_{n=1}^{\infty} b_n^2\right]$$

$$\int_0^L [f(x)]^2 dx = \frac{L}{2}\left[b_1^2 + b_2^2 + b_3^2 + \cdots\right]$$

(3) If $f(x)$ lies in the interval $(0, 2l)$ and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$\int_0^{2L} [f(x)]^2 dx = L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

(4) If $f(x)$ lies in $(c, c+2L)$

$$\frac{1}{2L}\int_c^{c+2L} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

5. Find FS of $f(x) = x^2$ in $(-\pi, \pi)$. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$f(x) = x^2$ is even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \underbrace{x^2}_{u} \underbrace{\cos nx \, dx}_{dv}$$

$du = 2x \qquad v = \frac{\sin nx}{n}$

$$a_n = \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} - \int \frac{2}{n} \underbrace{x}_{u} \underbrace{\sin nx \, dx}_{dv} \right]_0^{\pi}$$

$du = 1 \quad v = -\frac{\cos nx}{n}$

$$= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} - \frac{2}{n} \left[-\frac{x \cos nx}{n} + \int \frac{\cos nx}{n} \, dx \right] \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2}{n^2} x \cos nx - \frac{2}{n^3} \sin nx \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{2}{n^2}\cancel{\pi}\cos n\pi\right] = \frac{4}{n^2}(-1)^n$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

what we need: $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Parseval's Identity

$$2\int_0^{\pi} [f(x)]^2 dx = \pi\left[\sum_{n=1}^{\infty} a_n^2 + \frac{a_0^2}{2}\right]$$

$$2\int_0^{\pi} x^4 dx = \pi\left[\frac{\overset{2}{\cancel{4}}\pi^4}{9\times\cancel{2}} + \sum_{n=1}^{\infty} \frac{16}{n^4}\right]$$

$$2\frac{\pi^{\cancel{5}\,4}}{5} = \cancel{\pi}\left[\frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}\right]$$

$$\frac{2\pi^4}{16}\left(\frac{1}{5} - \frac{1}{9}\right) = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

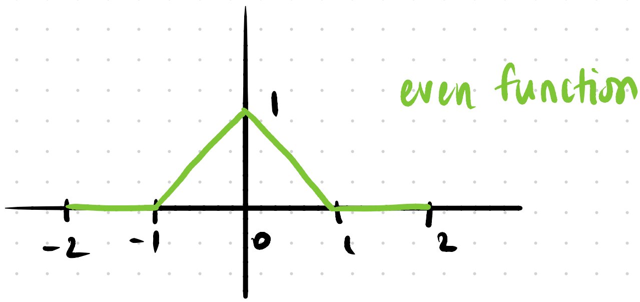
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{\cancel{8}\,2}\pi^4\left(\frac{\cancel{4}}{45}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

6. $f(t) = \begin{cases} 0 & ; \quad -2 < t \le -1 \\ 1+t & ; \quad -1 < t \le 0 \\ 1-t & ; \quad 0 < t \le 1 \\ 0 & ; \quad 1 < t < 2 \end{cases}$ Find FS

total interval: -2 to $+2$

$f(t)$ is an even function



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L}$$

$$a_0 = \frac{2}{2}\int_0^2 f(t)\,dt = \int_0^1 1-t\,dt + \int_1^2 0\,dt$$

$$= \left[t - \frac{t^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$a_0 = 1/2$

$$a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\,dt$$

$$= \int_0^1 (1-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \left[\frac{\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}} - \frac{t\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi t}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1$$

$$a_n = \left[\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} - \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} + \frac{1}{\left(\frac{n\pi}{2}\right)^2}\right]$$

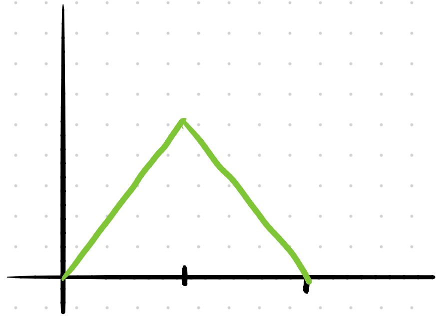
$$a_n = \frac{4}{(n\pi)^2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}\left(1-\cos\frac{n\pi}{2}\right)$$

7. Find FS for $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

$$f(x) = \pi x$$
$$f(2-x) = \pi(2-x)$$

even function



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{1}\int_0^1 \pi x \, dx = \left[\frac{2\pi x^2}{2}\right]_0^1 = \pi$$

$$a_n = \frac{2}{1}\int_0^1 \pi x \cos n\pi x \, dx$$

$$= 2\pi \left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1$$

$$= 2\pi \left[\frac{\cos n\pi}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right)$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}((-1)^n - 1)$$

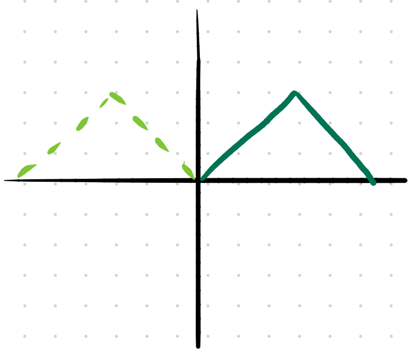
$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1)$$

Half Range Fourier Series

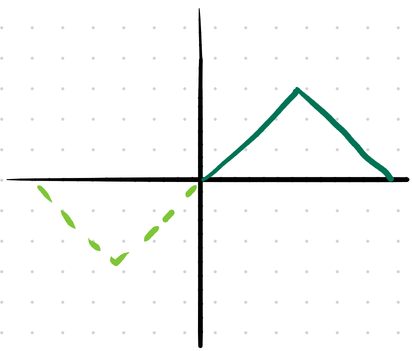
If $F(x)$ is defined in the interval $(0, \pi)$ and is not periodic in nature, we may expand the function to $f(x)$ and make it periodic with period 2π

It does not matter whether we make it even or odd as we will be interested in its value only from $(0, \pi)$

suppose $F(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$



even half-series (cosine)



odd half-series (sine)

8. Find half-range sine series for $F(x) = x^2$ in the interval $0 < x < 3$

Let $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ $\quad L = 3$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n x \pi}{3}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[\frac{-x^2\cos\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}} + \cancelto{0}{\frac{2x\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^2}} + \frac{2\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^3} \right]_0^3$$

$$= \frac{2}{3}\left[\frac{-9\cos n\pi}{\left(\frac{n\pi}{3}\right)} + \frac{2\cos n\pi}{\left(\frac{n\pi}{3}\right)^3} - \frac{2}{\left(\frac{n\pi}{3}\right)^2} \right]$$

$$= \frac{2}{3}\left[\frac{-27}{n\pi}(-1)^n + \frac{(27)(2)}{(n\pi)^3}\left((-1)^n - 1\right) \right]$$

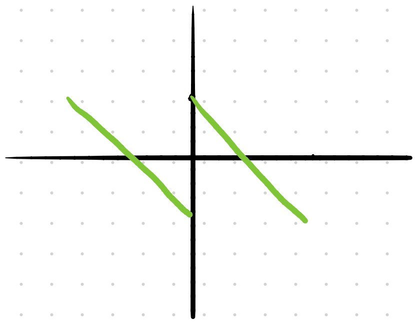
$$= 2\left[\frac{-9}{n\pi}(-1)^n + \frac{18}{(n\pi)^3}\left((-1)^n - 1\right) \right]$$

$$= \frac{-18}{n\pi}(-1)^n + \frac{36}{(n\pi)^3}\left((-1)^n - 1\right)$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{-18}{n\pi} (-1)^n + \frac{36}{(n\pi)^3} \left((-1)^n - 1\right) \right) \sin \frac{n\pi x}{3}$$

9. P.T $\frac{1}{2} - x = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{L}$, $0 < x < 1$

$$f(x) = \begin{cases} \frac{1}{2} - x & 0 < x < 1 \\ \frac{1}{2} - x & -1 < x \le 0 \end{cases}$$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1}$$

$$b_n = \frac{2}{1} \int_0^1 \left(\frac{1}{2} - x\right) \sin n\pi x \, dx$$

$$= 2 \left[\left(\frac{1}{2} - x\right) \frac{(-\cos n\pi x)}{n\pi} - \overset{0}{\cancel{\frac{(-1)(-\sin n\pi x)}{(n\pi)^2}}} \right]_0^1$$

$$= \cancel{2} \left[\left(+\frac{1}{\cancel{2}}\right) \frac{\cos n\pi}{n\pi} - \left(\frac{1}{\cancel{2}}\right) \frac{(-1)}{n\pi} \right]$$

$$= \frac{(-1)^n}{n\pi} \frac{+1}{n\pi} = \frac{1}{n\pi}\left((-1)^n + 1\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi}\left((-1)^n + 1\right) \sin n\pi x$$

10. Find the ½ range cosine series of $x\cos x$ $(0,\pi)$

$$f(x) = x\cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\cos x \, dx$$

$$= \frac{2}{\pi}\left[\cancel{x\sin x}^{\,0} - (1)(-\cos x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\cos\pi - \cos 0\right) = \frac{-4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos x\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left(\cos(n+1)x + \cos(n-1)x\right) dx$$

$$= \frac{1}{\pi}\left[\cancel{\frac{x\sin(n+1)x}{n+1}}^{\,0} + \frac{\cos(n+1)x}{(n+1)^2} + \cancel{\frac{x\sin(n-1)x}{n-1}}^{\,0} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{(n+1)^2} + \frac{\cos(n-1)\pi}{(n-1)^2} + \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} \right]$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{n+1}}{(n+1)^2} + \frac{(-1)^{n-1}}{(n-1)^2} + \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} \right)$$

$$= \frac{1}{\pi} \frac{(-1)^{n+1}+1}{(n+1)^2} + \frac{1}{\pi} \frac{(-1)^{n-1}+1}{(n-1)^2}$$

$$a_n = \frac{1}{\pi}\left(\frac{1-(-1)^n}{(n+1)^2} + \frac{1-(-1)^n}{(n-1)^2} \right) \qquad (n \neq 1)$$

for $n=1$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x\cos^2 x\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos^2 x\,dx$$

$$2a_1 = \frac{\pi}{\pi}\int_0^{\pi} \cos^2 x\,dx = 2\int_0^{\pi/2} \cos^2 x\,dx$$

$$2a_1 = \beta(1/2, 3/2)$$

$$a_1 = \frac{1}{2} \frac{\Gamma(1/2)(1/2)\Gamma(1/2)}{\Gamma(2)}$$

$$a_1 = \frac{\pi}{4}$$

$$f(x) = \frac{-2}{\pi} + \frac{\pi}{4}\cos x + \sum_{n=2}^{\infty}\left(\frac{1}{\pi}\frac{(1-(-1)^n)}{(n+1)^2} + \frac{1}{\pi}\frac{(1-(-1)^n)}{(n-1)^2}\right)\cos nx$$

11. Expand $\pi x - x^2$ in a half-range sine series in the interval $(0,\pi)$ upto first 3 terms

As a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[\cancelto{0}{\frac{(\pi x - x^2)(-\cos nx)}{n}} - \cancelto{0}{\frac{(\pi - 2x)(-\sin nx)}{n^2}} + \frac{(-2)(\cos nx)}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-2\cos n\pi}{n^3} + \frac{2}{n^3} \right]$$

$$= \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$$

$$\pi x - x^2 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1-(-1)^n\right)\sin nx$$

even terms $= 0$

odd terms $= \frac{2}{n^3}\sin nx$

$$= \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \dots\right)$$

12. Obtain sin half range series of f(x)

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x \le 1/2 \\ x - \frac{3}{4}, & 1/2 < x < 1 \end{cases} \qquad L = 1$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\int_0^{1/2} (1/4 - x) \sin n\pi x \, dx + 2\int_{1/2}^{1} (x - 3/4) \sin n\pi x \, dx$$

$$= 2\left[\left(\frac{1}{4} - x\right)\frac{(-\cos n\pi x)}{n\pi} - \frac{(-1)(-\sin n\pi x)}{(n\pi)^2} \right]_0^{1/2}$$

$$+ 2\left[\left(x - \frac{3}{4}\right)\frac{(-\cos n\pi x)}{n\pi} - \frac{(1)(-\sin n\pi x)}{(n\pi)^2} \right]_{1/2}^{1}$$

$$= 2\left[\cancel{\left(\frac{+1}{4}\right)\frac{(+\cos \frac{n\pi}{2})}{n\pi}}^{\,0} - \frac{(\sin \frac{n\pi}{2})}{(n\pi)^2} + \left(\frac{1}{4}\right)\frac{1}{n\pi} \right]$$

$$+ 2\left[\left(\frac{1}{4}\right)\frac{(-\cos n\pi)}{n\pi} - \cancel{\left(\frac{+1}{4}\right)\frac{(+\cos \frac{n\pi}{2})}{n\pi}}^{\,0} + \frac{(1)(-\sin \frac{n\pi}{2})}{(n\pi)^2} \right]$$

$$= 2\left[\frac{-2}{(\pi n)^2}\left(\sin\frac{n\pi}{2}\right) + \frac{1}{4}\left(\frac{1-(-1)^n}{n\pi}\right)\right]$$

HARMONIC ANALYSIS

If the function is not defined explicitly as a function of an independent variable but defined in terms of a graph or a table of values, then to find FS we perform Harmonic Analysis

Here, we cannot use Euler's formula to find a_0, a_n and b_n.

Direct current : $a_0/2$

First harmonic $a_1 \cos x + b_1 \sin x$

Second harmonic $a_2 \cos 2x + b_2 \sin 2x$

$\vdots$

The mean value of a function $y = f(x)$ over the range (a,b) is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

If a set of N values for a function $y = f(x)$ having 2π as a period at equidistant points of x is given in the interval $(c, c+2\pi)$, then the Fourier coefficients are:

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$= 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\,dx\right]$$

$= 2\,[\text{mean value of } y = f(x) \text{ in } (c, c+2\pi)]$

$$= 2\left[\frac{\Sigma y}{N}\right] = \frac{2}{N}\Sigma y$$

$$\boxed{a_0 = \frac{2}{N}\Sigma y}$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\cos nx\,dx\right]$$

$= 2\,[\text{mean value of } y\cos nx]$

$$\boxed{a_n = \frac{2}{N}\Sigma y\cos nx}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx = 2\,[\text{mean value of } y\sin nx]$$

$$\boxed{b_n = \frac{2}{N}\Sigma y\sin nx}$$

Note: If period $= 2L$,

$$a_n = \frac{2}{N}\Sigma y\cos\left(\frac{n\pi x}{L}\right) \qquad b_n = \frac{2}{N}\Sigma y\sin\left(\frac{n\pi x}{L}\right)$$

13. Expand y as FS upto first harmonic

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
y	7.9	7.2	3.6	0.5	0.9	6.8

period $= 2\pi$

$n = 6$

$a_0/2 = ?$

I harmonic $= a_1 \cos x + b_1 \sin x = ?$

$$y = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

$$a_0 = \frac{2}{N} \Sigma y = \frac{1}{3} \Sigma y$$

$$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{1}{3} \Sigma y \cos x$$

$$b_1 = \frac{2}{6} \Sigma y \sin x = \frac{1}{3} \Sigma y \sin x$$

x	y	$y\cos x$	$y\sin x$
0	7.9	7.9	0
$\pi/3$	7.2	3.6	6.26
$2\pi/3$	3.6	-1.8	3.12
π	0.5	-0.5	0
$4\pi/3$	0.9	-0.45	-0.78
$5\pi/3$	6.8	3.4	-5.89
	26.9	12.15	2.71

$a_0 = 8.97$

$a_1 = 4.05$

$b_1 = 0.90$

$$y = 4.4835 + 4.05\cos x + 0.90\sin x$$

14. Find coefficients of first 2 sin and cos terms for the data

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$
y	0	9.2	14.4	17.8	17.3	11.7

$2L = \pi \Rightarrow L = \pi/2$

$N = 6$

$$a_0 = \frac{2}{N}\sum y = \frac{\sum y}{3}$$

$$a_n = \frac{2}{N}\sum y \cos\frac{n\pi x}{L} = \frac{1}{3}\sum y \cos 2nx$$

$$b_n = \frac{2}{N}\sum y \sin\frac{n\pi x}{L} = \frac{1}{3}\sum y \sin 2nx$$

x	y	$y\cos 2x$	$y\sin 2x$	$y\cos 4x$	$y\sin 4x$
0	0	0	0	0	0
$\pi/6$	9.2	4.6	7.967	-4.6	7.967
$\pi/3$	14.4	-7.2	12.47	-7.2	-12.47
$\pi/2$	17.8	-17.8	0	17.8	0
$2\pi/3$	17.3	-8.65	-14.98	-8.65	14.98
$5\pi/6$	11.7	5.85	-10.13	-5.85	-10.13
	70.4	-23.2	-4.673	-8.5	0.347

$a_0 = 23.467 \Rightarrow a_0/2 = 11.73$

$a_1 = \frac{1}{3} \Sigma y \cos 2x = -7.73$

$b_1 = \frac{1}{3} \Sigma y \sin 2x = -1.56$

$a_2 = \frac{1}{3} \Sigma y \cos 4x = -2.83$

$b_2 = \frac{1}{3} \Sigma y \sin 4x = 0.115$

$f(x) = 11.73 - 7.73 \cos 2x - 1.56 \sin 2x - 2.83 \cos 4x$
$+ 0.115 \sin 4x$

15. Find first 3 coefficients of cosine and 2 coefficients of sine terms of FS for the following data

x	0	1	2	3	4	5
y	9	18	24	28	26	20

$N = 6$

$2L = 6 \Rightarrow L = 3$

$$a_0 = \frac{2}{N} \Sigma y \quad = \frac{1}{3} \Sigma y$$

$$a_n = \frac{2}{N} \Sigma y \cos \frac{n\pi x}{L} \quad = \frac{1}{3} \Sigma y \cos \frac{n\pi x}{3}$$

$$b_n = \frac{2}{N} \Sigma y \sin \frac{n\pi x}{L} \quad = \frac{1}{3} \Sigma y \sin \frac{n\pi x}{3}$$

$$a_1 = \frac{1}{3} \Sigma y \cos \frac{\pi x}{3}$$

$$a_2 = \frac{1}{3} \Sigma y \cos \frac{2\pi x}{3}$$

$$a_3 = \frac{1}{3} \Sigma y \cos \pi x$$

$$b_1 = \frac{1}{3} \Sigma y \sin \frac{\pi x}{3}$$

$$b_2 = \frac{1}{3} \Sigma y \sin \frac{2\pi x}{3}$$

x	y	$y\cos\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\cos\pi x$	$y\sin\frac{\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	9	9	9	9	0	0
1	18	9	-9	-18	15.59	15.59
2	24	-12	-12	24	20.78	-20.78
3	28	-28	28	-28	0	0
4	26	-13	-13	26	-22.52	22.52
5	20	10	-10	-20	-17.32	-17.32
	125	-25	-7	-7	-3.46	0

$a_0 = \frac{125}{3} \Rightarrow \frac{a_0}{2} = \frac{125}{6} = 20.83$

$a_1 = \frac{1}{3}(-25) = -8.33$ $\quad b_1 = \frac{-3.46}{3} = -1.15$

$a_2 = \frac{-7}{3} = -2.33$ $\quad b_2 = 0$

$a_3 = \frac{-7}{3} = -2.33$

$f(x) = 20.83 - 8.33\cos\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 2.33\cos\pi x - 1.15\sin\frac{\pi x}{3}$

II COMPLEX EXPONENTIAL SERIES

For a function $f(x)$ in the interval $(c, c+2L)$, Euler's identity can be modified to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}}{2} \right)$$

$$+ \sum_{n=1}^{\infty} b_n \left(\frac{e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}}{2i} \right)$$

$$f(x) = \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i} \right)}_{c_n} e^{\frac{in\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i} \right)}_{c_n} e^{\frac{-in\pi x}{L}}$$

Let $c_0 = \frac{a_0}{2}$

$c_n = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{a_n}{2} - \frac{ib_n}{2}$

Let $c_{-n} = \overline{c_n} = \frac{a_n}{2} + \frac{ib_n}{2}$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

contains c_n, c_0 and c_{-n}

we defined

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2L}\left[\int_c^{c+2L} \left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right) f(x)\,dx\right.$$

$$c_n = \frac{1}{2L}\int_c^{c+2L} e^{\frac{-in\pi x}{L}} f(x)\,dx$$

complex exponential form

- used in celestial motion, signals

Note: only 1 single constant used (instead of 3)

16. Find complex FS for $f(x) = e^{-x}$ defined in $(-1,1)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$L = 1$

$$C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x(1+in\pi)}\, dx$$

$$C_n = \frac{1}{2}\left[\frac{e^{-x(1+in\pi)}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)}}{-(1+in\pi)} + \frac{e^{(1+in\pi)}}{(1+in\pi)}\right]$$

$$= \frac{1}{2}\left(\frac{e\, e^{in\pi} - e^{-in\pi} e^{-1}}{(1+in\pi)}\right)$$

$$= \frac{1}{2}\left(\frac{e(\cos n\pi + i\sin n\pi) - \frac{1}{e}(\cos n\pi - i\sin n\pi)}{(1+in\pi)}\right)$$

$$= \frac{1}{2}\left((e - \tfrac{1}{e})\frac{\cos n\pi}{1+in\pi}\right)$$

$$= \frac{1}{2}(e - e^{-1})\left(\frac{(-1)^n}{1+(n\pi)^2}(1-in\pi)\right)$$

$$c_n = \sinh(1)\, \frac{(-1)^n\, (1-in\pi)}{1+(n\pi)^2}$$

$$f(x) = \sum_{n=-\infty}^{\infty} e^{in\pi x} \frac{\sinh(1)\ (-1)^n\ (1-in\pi)}{1+(n\pi)^2}$$

17. Obtain complex FS expansion for $f(x) = \cos ax$ in $(-\pi, \pi)$

$L = \pi$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{in\pi x}{L}} = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax \, e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \cos ax \, dx$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{(-in)^2 + a^2} (-in \cos ax + a \sin ax) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{a^2-n^2}(a\sin ax - in\cos ax)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{e^{-in\pi}}{a^2-n^2}(a\sin a\pi - in\cos a\pi) - \frac{e^{in\pi}}{a^2-n^2}(-a\sin a\pi - in\cos a\pi)\right)$$

$$= \frac{1}{\pi}\frac{1}{a^2-n^2}(in\cos a\pi)\left(\frac{e^{in\pi}-e^{-in\pi}}{2}\right) + \frac{1}{\pi}\frac{1}{a^2-n^2}(a\sin a\pi)\left(\frac{e^{in\pi}+e^{-in\pi}}{2}\right)$$

$$= \frac{1}{\pi(a^2-n^2)}\left[\overbrace{(in\cos a\pi)\sin n\pi}^{0} + a\sin a\pi\cos n\pi\right]$$

$$c_n = \frac{a\sin a\pi\,(-1)^n}{\pi(a^2-n^2)}$$

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{a\sin a\pi}{\pi(a^2-n^2)}(-1)^n e^{inx}$$

Circuit Application of Fourier Series

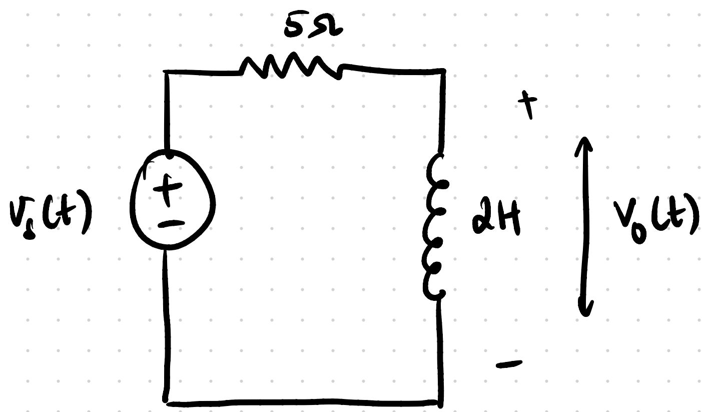
To find steady state response of a circuit.

The circuit shown has a non-sinusoidal $v_s(t)$ source that has a Fourier series

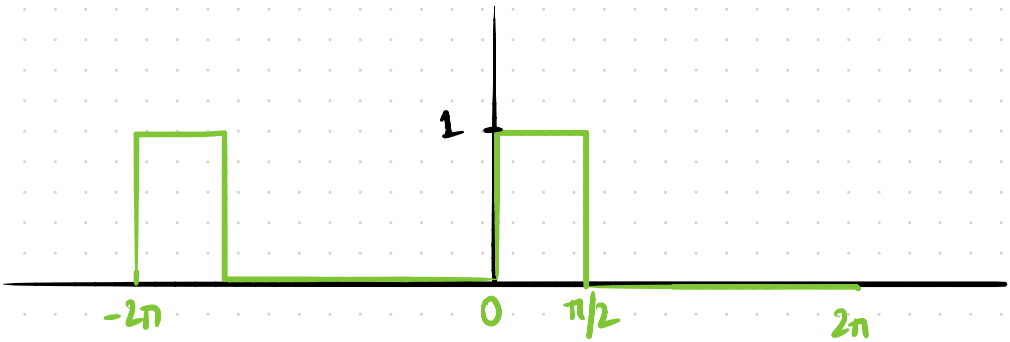
$$v_s(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{n} \sin(n\pi t)$$

for $n = 2k-1$

Find the voltage $v_o(t)$ at inductor and the corresponding amplitude spectrum



18. Find FS expansion of the pulse train function as shown



$$f(x) = \begin{cases} 1, & 0 \leq x < \pi/2 \\ 0, & \pi/2 < x \leq 2\pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi/2} dx = \frac{1}{2}$$

$$\frac{a_0}{2} = \frac{1}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos nx \, dx = \left[\frac{\sin nx}{n\pi}\right]_0^{\pi/2}$$

$$a_n = \frac{\sin \frac{n\pi}{2}}{n\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx = \frac{1}{\pi}\int_0^{\pi/2} \sin nx \, dx$$

$$b_n = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} = \frac{1}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}\cos nx}{n}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)\sin nx$$